How to Measure the Earth's Size by Taking a Photo?

Reporter: Yuan-Pei Yang (杨元培)

Reference: Robert J. Vanderbei, The earth is not flat: An analysis of a sunset photo, Optics & Photonics News (OPN), 34-39, November 2008

2024/12, Journal Club @ SWIFAR, Yunnan University

Earth is Not Flat

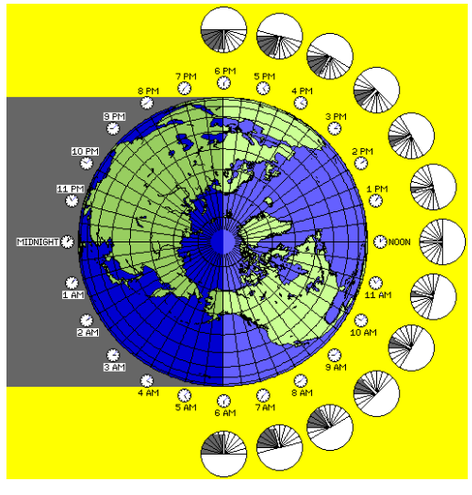
- There are several well-known ways to prove to yourself that the ***Earth is round*** by
  - looking earth on moon
  - looking lunar eclipse
  - watching ships disappear over the horizon,
  - noting the presence of time zones
  - … …

Looking Earth

Lunar Eclipse

Disappearing Ship

Time Zones



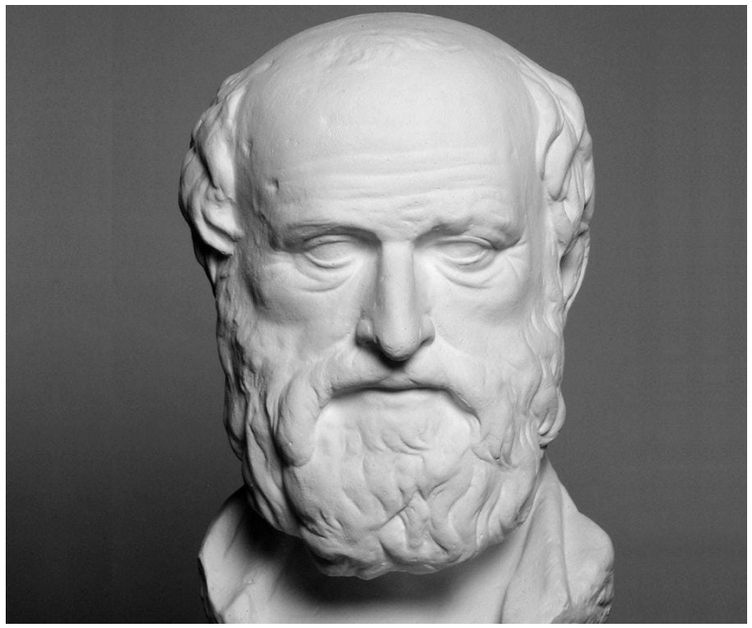
Eratosthenes' Measurement

- At around 200 BC, **Eratosthenes** (c.276 BC - c.197 BC) determined the circumference of Earth with remarkable accuracy. As related accomplishments, he also demonstrated that Earth is essentially spherical and measured the tilt of its axis (23.5°).

- Eratosthenes compared the midsummer noon-time shadow at **Syene** (now Aswan, southern Egypt) to that at **Alexandria** (Mediterranean coast, northern Egypt).

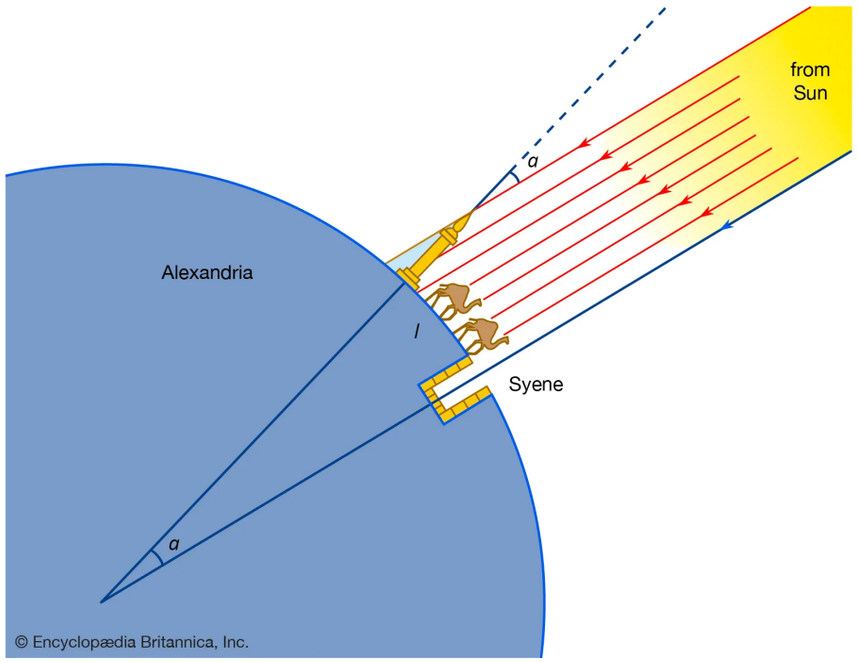
- **At Syene (23.5°north latitude), he observed that at solar noon the sun was directly overhead, i.e. perpendicular to Earth's surface.**

- **At the same time in Alexandria the sun was 82.8° above the southern horizon or, in other words, 7.2° from a point directly overhead.**



Eratosthenes

Egypt Map



Shadow Direction

Eratosthenes' Measurement

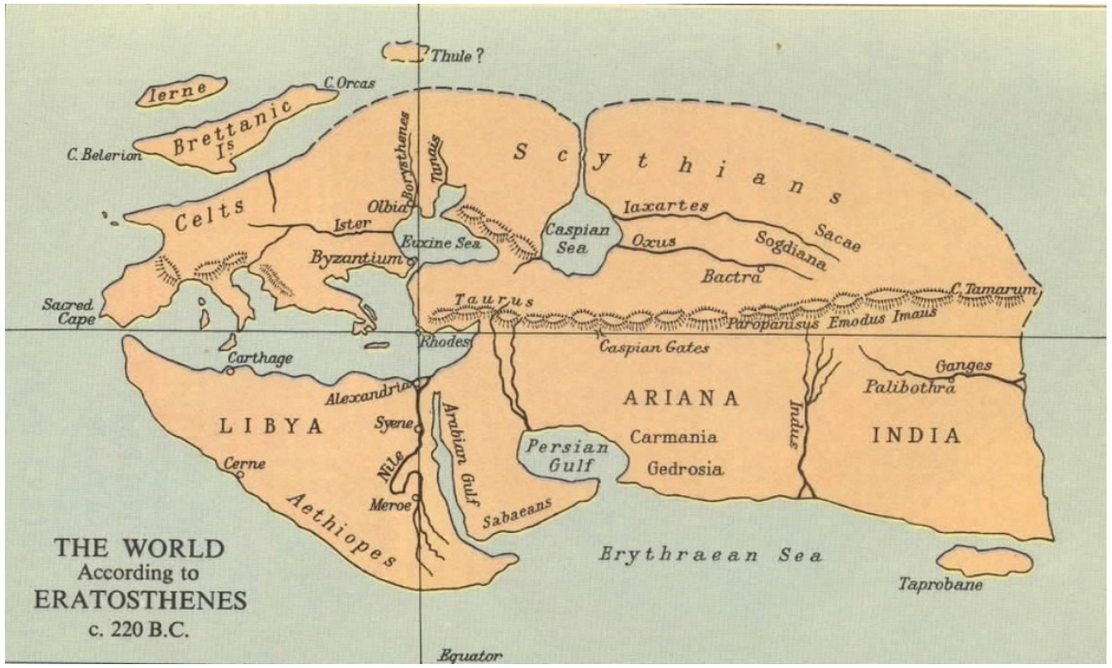
- Eratosthenes knew Alexandria to be **5,000 stadia north of Syene**. (The stadia is an ancient unit of length equal to 160m).

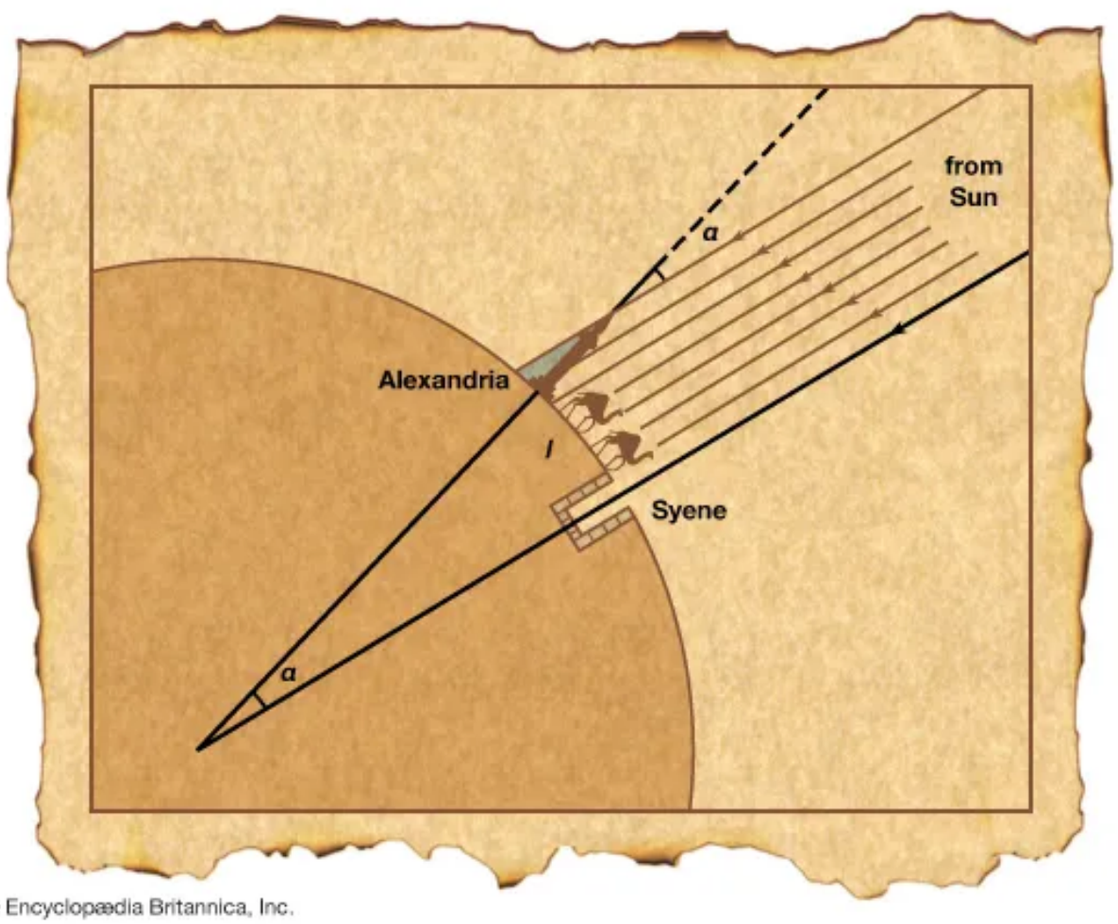
- He figured that **7.2° is 1/50th of 360° and argued that Alexandria**, therefore, must be 1/50th of the way around the spherical Earth from Syene.

- He calculated the circumference of Earth as 50 times the distance between Syene and Alexandria and found it to be **250,000 stadia or about 25,000 miles (40000 km)**.

- Alexandria is not directly north of Syene, **they have a difference of 3 degrees in longitude**.

- At noon on the summer solstice, **the sun does not fully shine directly on Syene, with a deviation of 0.4°**. These two errors just are canceled each other.





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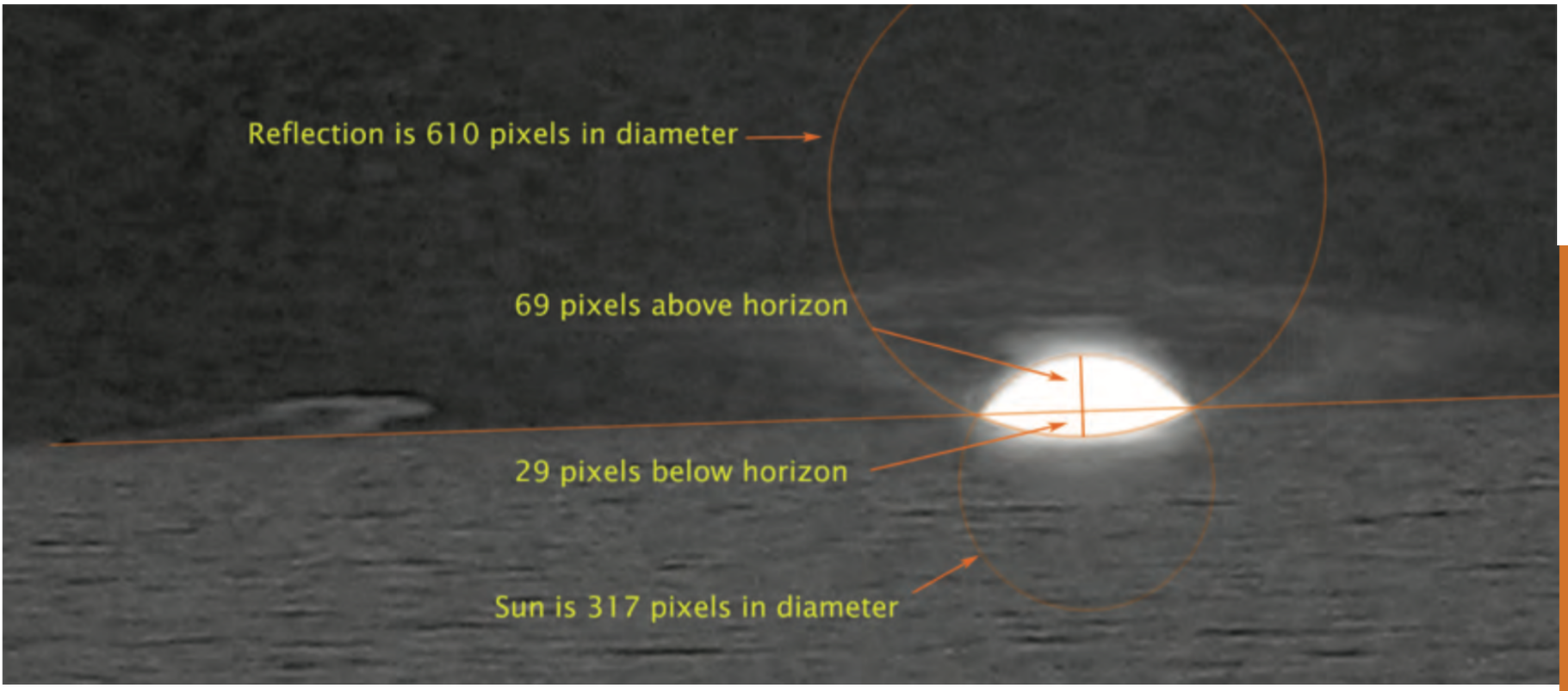
Is there a way to measure the Earth's size at only one location?

Just Need Take a Sunset Photo near a Lake!

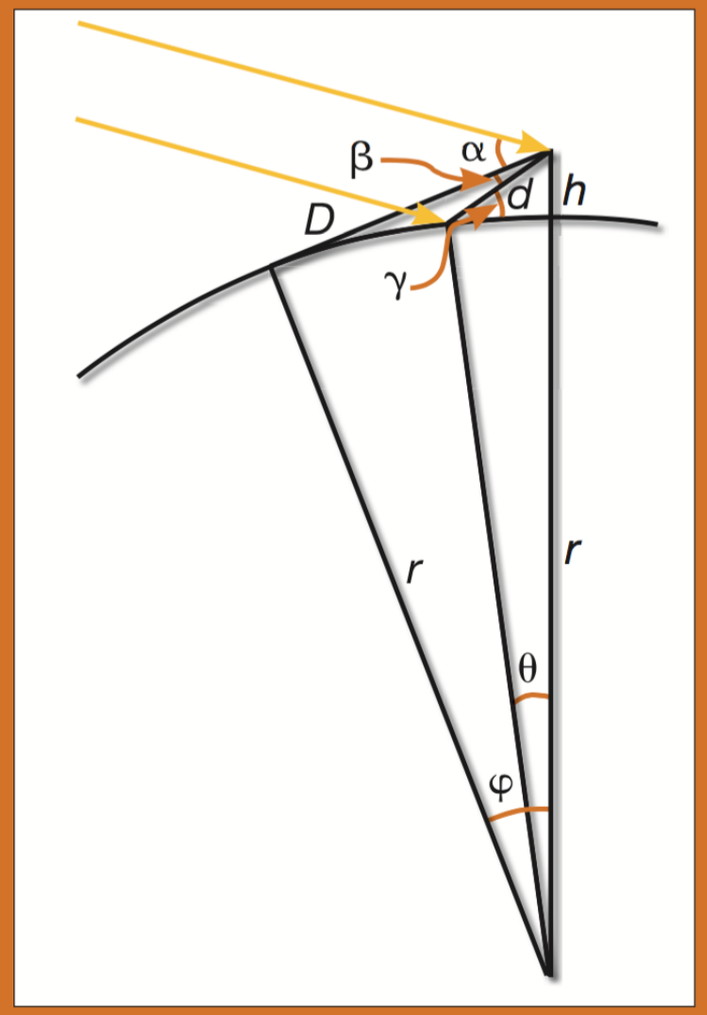
The sunset image was taken over Lake Michigan on July 5, 2008. by Prof. Robert J. Vanderbei from Princeton University

Not Need on Moon!

Vanderlei's Measurement



$$\alpha = (69/317) \times 0.5°$$
$$\beta = (29/317) \times 0.5°$$



The paths of two sun rays coming from the upper limb of the sun. One ray bounces off the water and into the camera; the other goes straight to the camera.

- If the Earth were at, the water calm, and the air clear, the Sun would make *a perfect mirror image reflection* on the water.

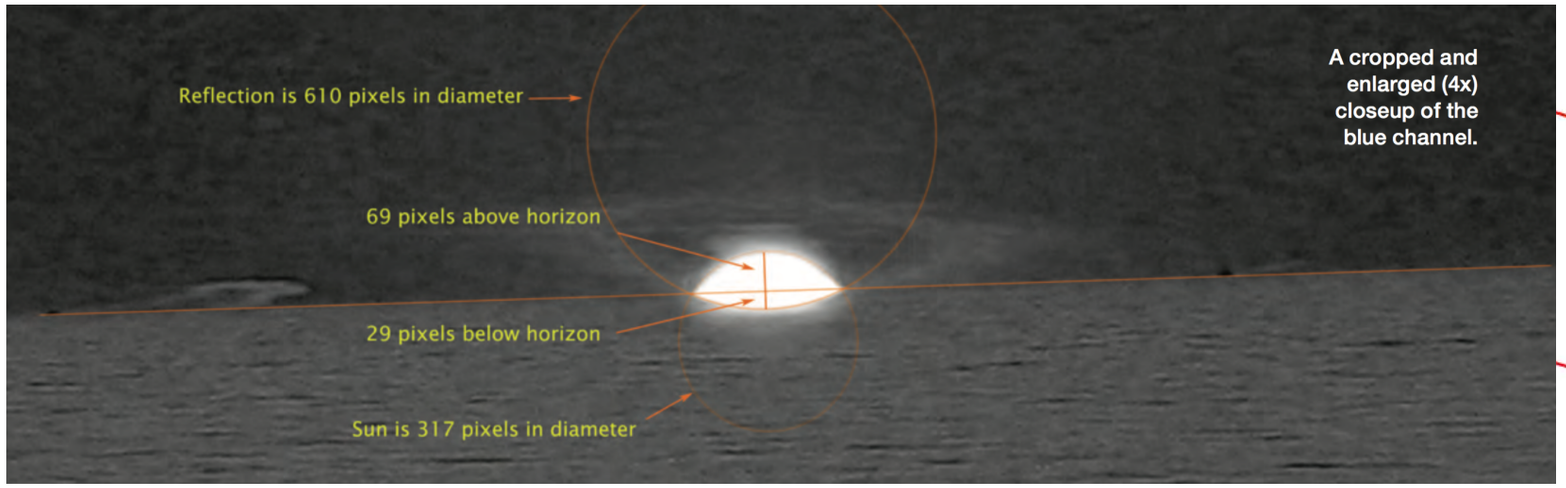
- The cropped close-up shows *a reflection that is clearly foreshortened*.

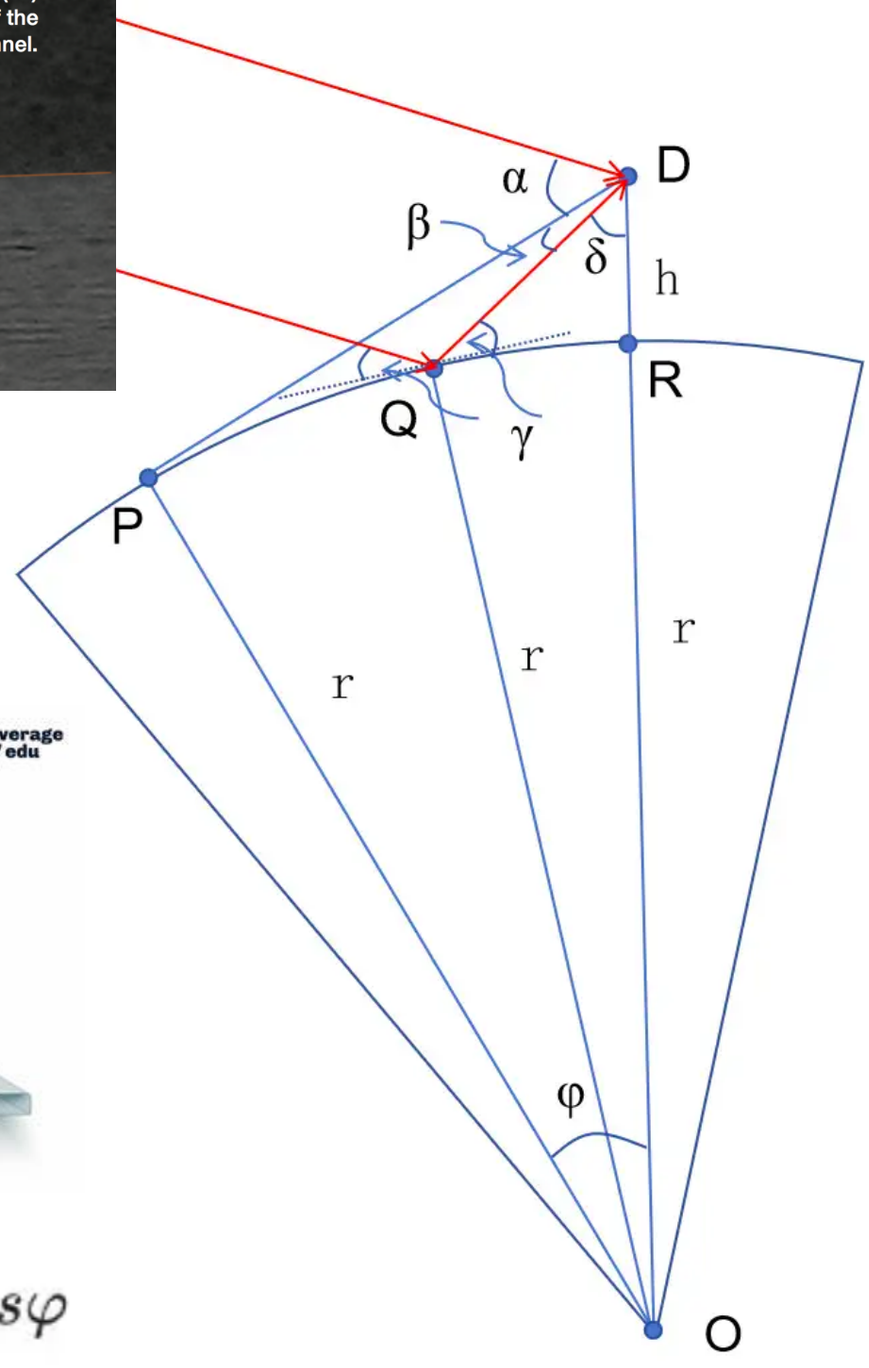
Basic two principles:

1. **Light propagates in a straight line;**

2. **The incident angle is equal to the reflection angle.**

Vanderlei's Measurement



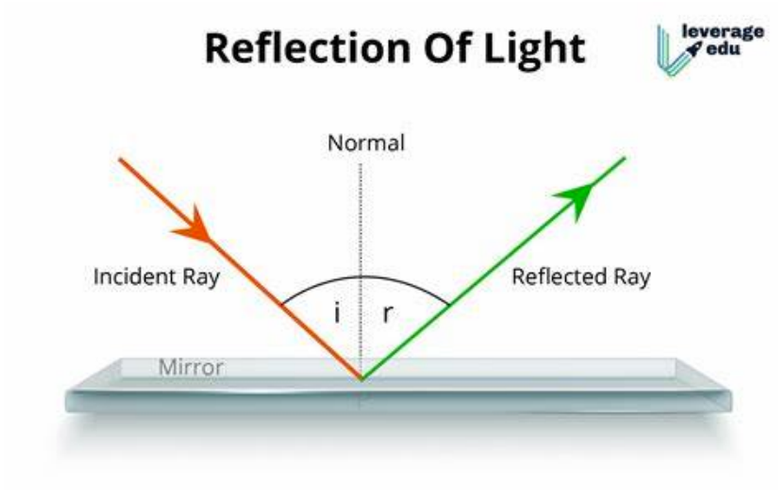


- According to **the reflection law (as the only physical law in this method)**

$$2\gamma = \alpha + \beta$$

- △OPD is a right triangle

$$\varphi + \beta + \delta = \frac{\pi}{2}$$



- In △OQD and △OPD, one has

$$\frac{r}{sin\left(\frac{\pi}{2}+\gamma\right)} = \frac{r+h}{sin\delta} \qquad \frac{r}{r+h} = sin(\delta+\beta) = cos\varphi$$

-

Vanderlei's Measurement

- Based on the above two equation, one has

$$cos\varphi = \frac{sin\delta}{sin\left(\frac{\pi}{2}+\gamma\right)} = \frac{cos(\varphi+\beta)}{cos\gamma}$$

- leading to

$$tan\varphi = \frac{cos\beta - cos\gamma}{sin\beta} = \frac{2sin\frac{\alpha-\beta}{4}sin\frac{\alpha+3\beta}{4}}{sin\beta}$$

$$r = \frac{hcos\varphi}{1-cos\varphi}$$

Earth radius as function of α, β and h

- **In small-angle approximation,** one finally has

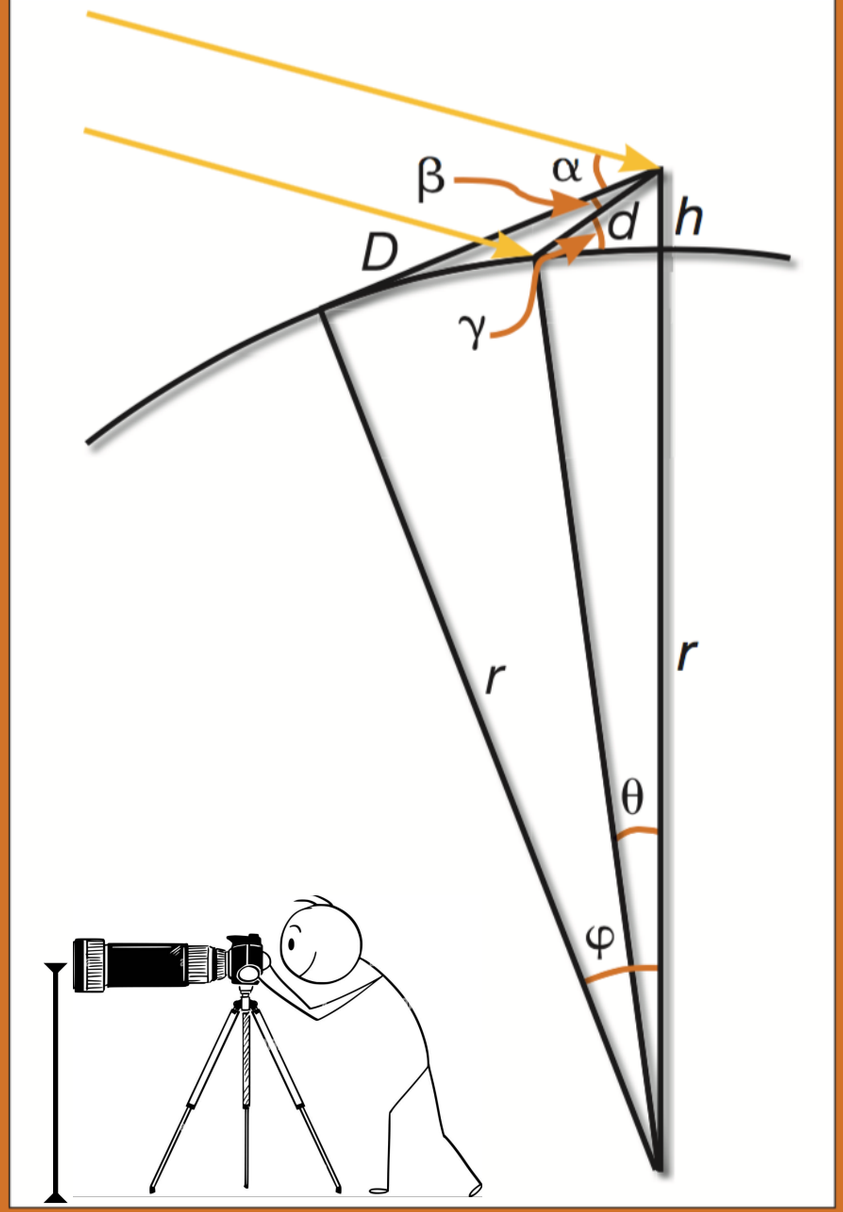
$$r = \frac{2h}{\varphi^2} \qquad \varphi = \frac{(\alpha-\beta)(\alpha+3\beta)}{8\beta}$$

- Plugging in the numbers

$$\alpha = (69/317) \times 0.5°$$
$$\beta = (29/317) \times 0.5°$$

Camera Height: h ~ 1.75m (estimation)

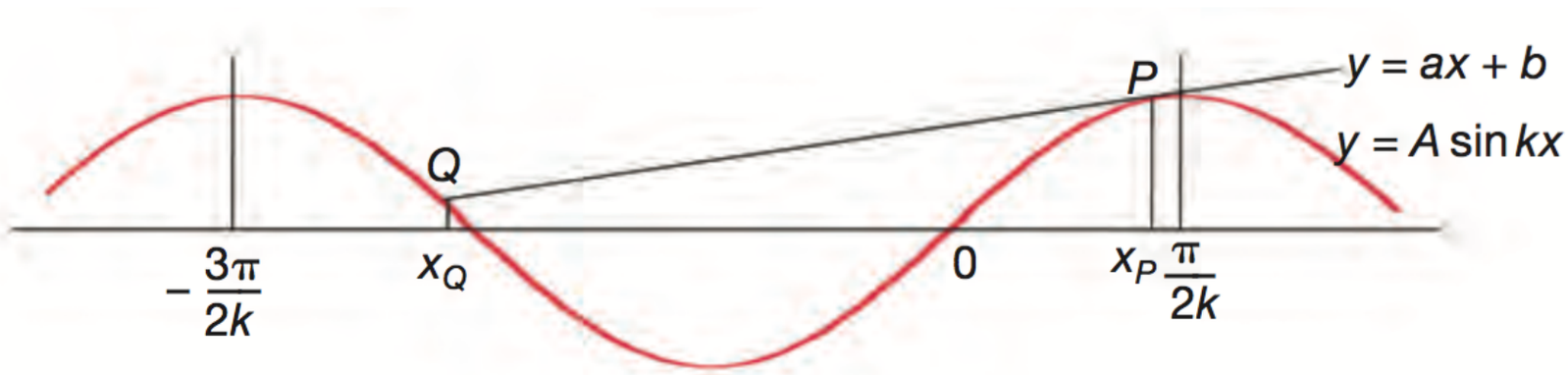
- **One finally obtains r = 6384 km, which is very close to the real value of 6400km**



The paths of two sun rays coming from the upper limb of the sun. One ray bounces off the water and into the camera; the other goes straight to the camera.

The Effect of Waves

- Normally, waves wreak havoc on sunset reflections. Certainly Lake Michigan and the other Great Lakes are calmer than the oceans.



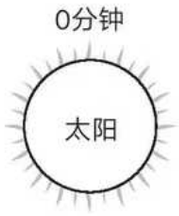
$$y'(x_Q) = Ak \cos kx_Q$$
$$= -\sqrt{2\pi aAk}$$
$$y'(x_P) = a.$$

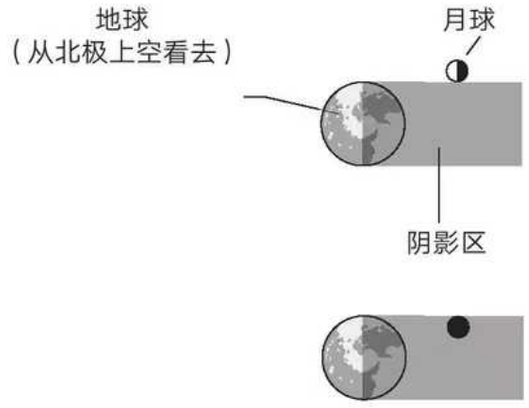
Diagram illustrating the paths of a sun ray at a certain inclination.

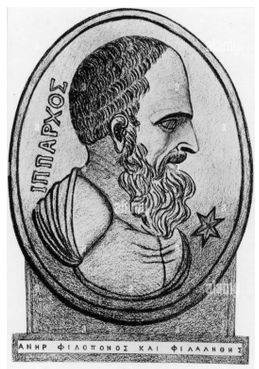
- **Tall waves:**
  - Assume that the maximum slope of the waves is greater than the angle of the sun ray $\frac{a}{Ak} \leq 1$
  - The range of upward reflection angles varies from essentially zero to $\text{max angle} = a + 2\sqrt{2\pi aAk}$.
- **Small waves**
  - The incoming light beam illuminates all parts of the wave.
  - The greatest possible angle of the reflected beam is $a + 2Ak$.
  - **If *Ak* is much smaller than *a*, then this reflected angle well approximates a itself.**

Building the Universe by Ancient Greeks

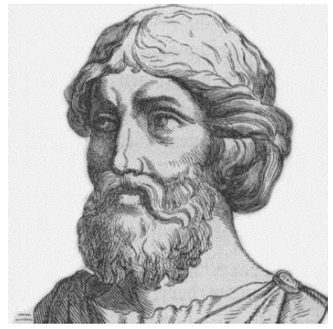
Moon' diameter by *Hipparchus*

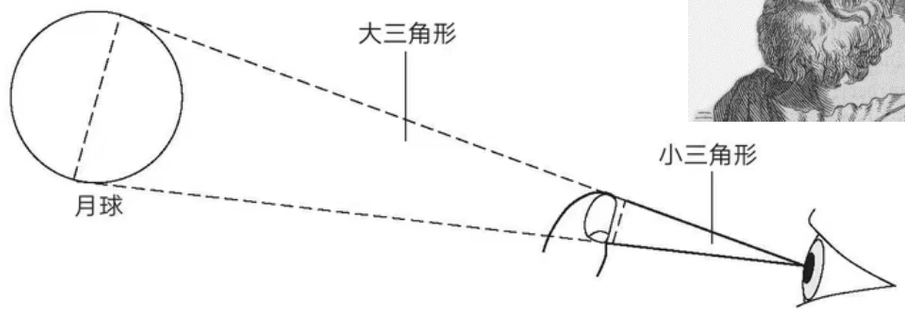


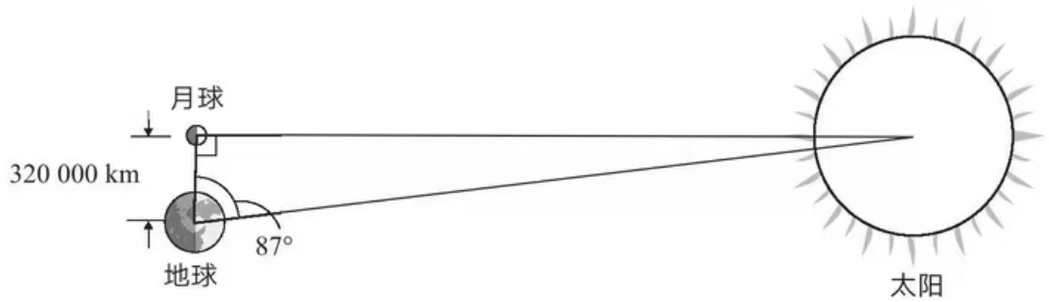


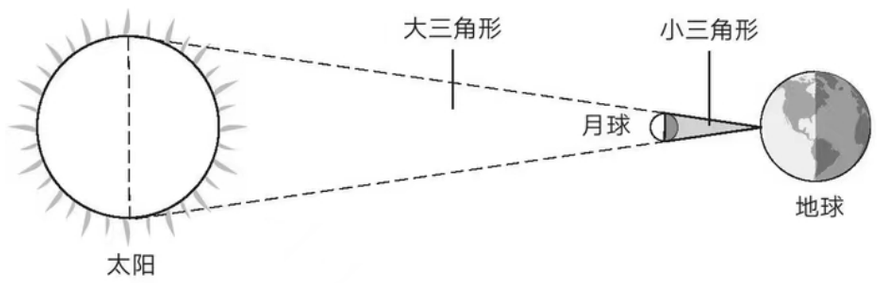


Earth-Moon distance, Earth-Sun distance, Sun's diameter by *Aristarchus*

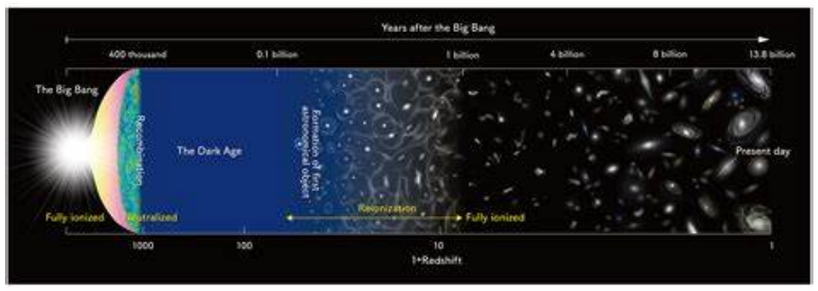








Baby Universe vs Today's Universe →



Earth's diameter is the starting point!

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Thanks for Your Patience !
And this is the real first page

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